

Non-escaping endpoints do not explode

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ABSTRACT

The family of exponential maps $f_a(z) = e^z + a$ is of fundamental importance in the study of transcendental dynamics. Here we consider the topological structure of certain subsets of the Julia set $J(f_a)$. When $a \in (-\infty, -1)$, and more generally when a belongs to the Fatou set $F(f_a)$, it is known that $J(f_a)$ can be written as a union of *hairs* and *endpoints* of these hairs. In 1990, Mayer proved for $a \in (-\infty, -1)$ that, while the set of endpoints is totally separated, its union with infinity is a connected set. Recently, Alhabib and the second author extended this result to the case where $a \in F(f_a)$, and showed that it holds even for the smaller set of all *escaping* endpoints.

We show that, in contrast, the set of *non-escaping* endpoints together with infinity is totally separated. It turns out that this property is closely related to a topological structure known as a ‘spider’s web’; in particular we give a new topological characterisation of spiders’ webs that may be of independent interest. We also show how our results can be applied to *Fatou’s function*, $z \mapsto z + 1 + e^{-z}$.

1. Introduction

The iteration of transcendental entire functions was initiated by Fatou in 1926 [13], and has received considerable attention in recent years. The best-studied examples are provided by the functions

$$f_a: \mathbb{C} \rightarrow \mathbb{C}; \quad z \mapsto e^z + a \tag{1.1}$$

for $a \in (-\infty, -1)$, see [1, 9, 11, 15]. In this case, there is a unique attracting fixed point ζ on the negative real axis. The (open) set of starting values whose orbits under f_a converge to ζ under iteration is connected and dense in the plane (see Figure 1).

The complement of this basin of attraction, the *Julia set* $J(f_a)$, is known to be an uncountable union of pairwise disjoint arcs, known as ‘hairs’, each of which joins a finite *endpoint* to infinity. More precisely, $J(f_a)$ is a ‘Cantor bouquet’; see [1, 4] for further information. The action of f_a on $J(f_a)$ provides the simplest (yet far from simple) transcendental entire dynamical system. Results first established in this context have often led to an increased understanding in far more general settings.

A topological model of $f_a|_{J(f_a)}$ was given in [1] in terms of a *straight brush*, where the hairs of $J(f_a)$ are represented by straight horizontal rays. This model depended *a priori* on the parameter $a \in (-\infty, -1)$, but a natural version of the construction that is independent of a was given in [22]. From this point of view, the topological dynamics in this case is completely understood, but the set $J(f_a)$ nonetheless exhibits a number of subtle and surprising properties (compare [15] for a celebrated and unexpected result concerning Hausdorff dimensions).

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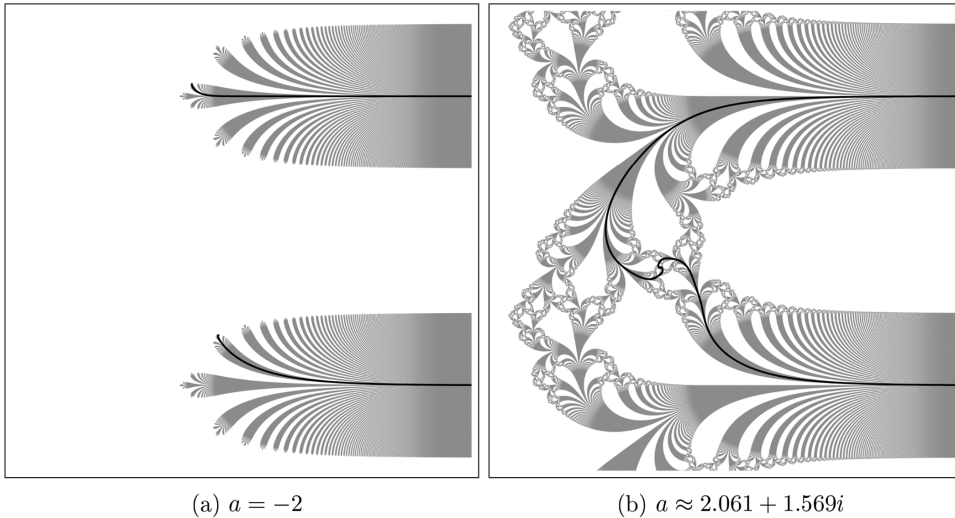


FIGURE 1. Two maps in the exponential family; the Julia set is shown in grey, while in each picture two individual hairs are shown in black. The white regions correspond to points which converge to an attracting periodic cycle under iteration. In (a), the Julia set is a Cantor bouquet, and different hairs have different endpoints. The map in (b) has an attracting cycle of period 3; several hairs share the same endpoint, as is the case for the two hairs shown.

In particular, Mayer [17] proved in 1990 that the set $E(f_a)$ of endpoints of f_a (see above) has the intriguing property that $E(f_a) \cup \{\infty\}$ is connected, while $E(f_a)$ itself is totally separated. Here a totally separated space is defined as follows.

DEFINITION 1.1 (Separation). Let X be a topological space. Two points $a, b \in X$ are *separated* (in X) if there is an open and closed subset $U \subset X$ with $a \in U$ and $b \notin U$. If every pair of points in X is separated, we say that X is a *totally separated* space.

If X is connected but $X \setminus \{x_0\}$, $x_0 \in X$, is totally separated, then we say that x_0 is an *explosion point* of X . Hence, infinity is an explosion point for $E(f_a) \cup \{\infty\}$, $a < -1$. Following the terminology used in [2], we will also simply say that infinity is an explosion point for $E(f_a)$.

Alhabib and the second author recently proved [2, Theorem 1.3] that Mayer's result holds also for the smaller set of *escaping endpoints* of $J(f_a)$; that is, for the set $\tilde{E}(f_a) := E(f_a) \cap I(f_a)$ of endpoints that belong to the escaping set

$$I(f_a) = \{z \in \mathbb{C} : f_a^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

Still assuming that $a \in (-\infty, -1)$, the complementary set of *non-escaping endpoints* of f_a satisfies the following identities:

$$E(f_a) \setminus \tilde{E}(f_a) = J(f_a) \setminus I(f_a) = J_r(f_a);$$

see Corollary 2.2 and Proposition 2.4. Here $J_r(f_a)$ is the *radial Julia set*, a set of particular importance. The results of [2] naturally suggest the question whether ∞ is an explosion point for $J_r(f_a)$ also. It is known [36, Section 2] that $J_r(f_a)$ has Hausdorff dimension strictly greater than one, which is compatible with this possibility. Nonetheless, we prove here that the sets of escaping and non-escaping endpoints are topologically very different from each other.

THEOREM 1.2 (Non-escaping endpoints do not explode). *Let $a \in (-\infty, -1)$. Then the set $J_r(f_a) \cup \{\infty\}$ is totally separated.*

The maps f_a as in (1.1), for general $a \in \mathbb{C}$, have also been investigated in considerable detail; compare, for example, [3, 11, 23, 28, 33]. This family can be considered as an analogue of the family of quadratic polynomials, which gives rise to the famous Mandelbrot set (see [8, 27]).

The dynamical structure of f_a for general a is much more complicated than for real $a < -1$, but nonetheless Theorem 1.2 can be extended to a large and conjecturally dense set of parameters. More precisely, it is known [23] that the *Julia set* $J(f_a)$ (informally, the locus of chaotic dynamics; see Section 2) can be written as a union of hairs and endpoints if and only if $a \notin J(f_a)$. In this case, a belongs to the *Fatou set* $F(f_a) = \mathbb{C} \setminus J(f_a)$, and $F(f_a)$ is precisely the basin of attraction of an attracting or parabolic periodic cycle (see Proposition 2.1). Note that $F(f_a)$ is no longer connected in general; compare Figure 1(b). In particular, $J(f_a)$ is not a Cantor bouquet in this case, but rather a more complicated structure where different curves share the same endpoint, known as a *pinched Cantor bouquet*. (For further details, see Proposition 2.4 and its proof.)

All of the above discussion for the case $a < -1$ carries over to the case where $a \in F(f_a)$, with the exception that the radial Julia set is a proper subset of the set of non-escaping points when $F(f_a)$ has a parabolic cycle. By [2], the set $\tilde{E}(f_a)$ has ∞ as an explosion point; in contrast, we show that $(J(f_a) \setminus I(f_a)) \cup \{\infty\}$ is totally separated also in this case.

We can further strengthen Theorem 1.2 by considering the *speed* with which points of $\tilde{E}(f_a)$ escape to infinity. The *fast escaping set* $A(f)$ of a transcendental entire function f , introduced by Bergweiler and Hinkkanen [6] and investigated closely by Rippon and Stallard, see [29, 31], has played an important role in recent progress in transcendental dynamics. Informally, $A(f)$ consists of those points of $I(f)$ that tend to infinity at the fastest rate possible; for a formal definition, see (2.2). Let us say that a point $z \in J(f)$ is *meandering* if it does not belong to $A(f)$, and denote the set of meandering points by $J_m(f)$. It is known that for f_a , $a \in \mathbb{C}$, every point on a hair belongs to $A(f_a)$ (see [33, Lemma 5.1] and compare also [26]); in particular, when $a \in F(f_a)$ every meandering point is an endpoint.

By [2, Remark on p. 68], for a as above, infinity is an explosion point even for the set $E(f_a) \cap A(f_a)$ of fast escaping endpoints. In contrast, we show the following.

THEOREM 1.3 (Meandering endpoints do not explode). *Suppose that $a \in \mathbb{C}$ is such that $a \in F(f_a)$. Then the set $J_m(f_a) \cup \{\infty\}$ is totally separated.*

Spiders' webs

Our results have a connection with a topological structure introduced by Rippon and Stallard in [31], known as a 'spider's web'.

DEFINITION 1.4 (Spider's web). A set $E \subset \mathbb{C}$ is an (*infinite*) *spider's web* if E is connected and there exists a sequence $(G_n)_{n=0}^\infty$ of bounded simply connected domains, with $G_n \subset G_{n+1}$ and $\partial G_n \subset E$ for $n \geq 0$, such that $\bigcup_{n=0}^\infty G_n = \mathbb{C}$.

We will prove that the complement $A(f_a) \cup F(f_a)$ of $J_m(f_a)$ is a spider's web when $a \in F(f_a)$. Together with the fact that $J_m(f_a) \subset E(f_a)$ itself is totally separated (in \mathbb{C}), this easily implies Theorem 1.3. Our proof uses a new topological characterisation of spiders' webs, which may have independent interest.

THEOREM 1.5 (Characterisation of spiders' webs). *Let $E \subset \mathbb{C}$ be connected. Then E is a spider's web if and only if E separates every finite point $z \in \mathbb{C}$ from ∞ .*

(Here E separates z from ∞ if the two points are separated in $(\mathbb{C} \setminus E) \cup \{z, \infty\}$.)

Singular values in the Julia set

As mentioned above, when $a \in J(f_a)$, it is no longer possible to write $J(f_a)$ as a union of hairs and endpoints, so questions concerning the structure of the set of endpoints are less natural in this setting. (The set of *escaping endpoints*, on the other hand, does remain a natural object; compare the discussion in [2].)

However, the radial Julia set, the set of non-escaping points and the set of meandering points remain of interest, and it turns out that their structure usually differs dramatically from the case where $a \in F(f_a)$. Indeed, consider the *postsingularly finite* case, where the omitted value a eventually maps onto a repelling periodic cycle. Then $J(f_a) = \mathbb{C}$, and $J_r(f_a) = \mathbb{C} \setminus I(f_a)$ (see Corollary 2.2(c)). It follows from results of [23] that $J_r(f_a)$ contains a dense collection of unbounded connected sets. In particular, $J_r(f_a) \cup \{\infty\}$ is connected, and $I(f_a)$ is not a spider's web (see Proposition 2.4 and Corollary 4.3). It is plausible that $(\mathbb{C} \setminus I(f_a)) \cup \{\infty\}$ is connected for all $a \in \mathbb{C}$, and that, in particular, $I(f_a)$ and $A(f_a)$ are never spiders' webs.

The techniques from our proof of Theorem 1.3 can nonetheless be adapted to yield a slightly technical result about the set of meandering points whose orbits stay away from the singular value (Theorem 4.1). In particular, we recover a result from [21], concerning a question of Herman, Baker and Rippon about the boundedness of certain Siegel discs in the exponential family (Theorem 4.2).

We leave open the question whether, when $a \in J(f_a)$, infinity can be an explosion point for the set $J_m(f_a) \cap I(f_a)$ of points that are meandering and escaping.

Fatou's function

Our proofs build on an idea from recent work of the first author [12] concerning Fatou's function

$$f: \mathbb{C} \rightarrow \mathbb{C}; \quad z \mapsto z + 1 + e^{-z}.$$

For this function, the Julia set is once again a Cantor bouquet, but in contrast to the exponential family the Fatou set here is contained in the escaping set $I(f)$. It is shown in [12] that $I(f)$ is a spider's web, and that the set of non-escaping endpoints of f together with infinity is *totally disconnected*. We will show that Theorem 1.2 implies the stronger result that $A(f) \cup F(f)$ is a spider's web for Fatou's function, and that the set of meandering endpoints $J_m(f) \subset J(f)$ together with infinity is a totally separated set. In particular, this illustrates that our results have consequences beyond the exponential family itself.

Structure of the article

In Section 2, we review key definitions and facts from exponential dynamics. We also establish Theorem 1.5, concerning spiders' webs. Our results about exponential maps f_a with $a \in F(f_a)$ are proved in Section 3, while the case where $a \in J(f_a)$ is discussed in Section 4. Finally, we consider Fatou's function in Section 5.

2. Preliminaries

Notation and background

We denote the complex plane by \mathbb{C} and the Riemann sphere by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The round disc of radius r around a point z_0 is denoted by $D(z_0, r)$.

Let f be a transcendental entire function. As noted in the introduction, the *Julia set* and *Fatou set* of f are denoted $J(f)$ and $F(f)$, respectively. Here $F(f)$ is the set of normality of the

family of iterates $(f^n)_{n=1}^\infty$ of f , and $J(f) = \mathbb{C} \setminus F(f)$ is its complement. For further background on transcendental dynamics, we refer to [5].

The *radial Julia set* $J_r(f) \subset J(f)$ can be described as the locus of non-uniform hyperbolicity: at a point $z \in J_r(f)$, it is possible to pass from arbitrarily small scales to a definite scale using univalent iterates. More formally, there is a number $\delta > 0$ and infinitely many n such that the spherical disc of radius δ around $f^n(z)$ can be pulled back univalently along the orbit. For rational functions, the radial Julia set first appeared implicitly in work of Lyubich [16], and was introduced formally by Urbński [35] and McMullen [19]. As far as we are aware, its first appearance in the entire setting, in the special case of hyperbolic exponential maps, is due to Urbński and Zdunik [36]. We refer to [25] for a general discussion. In the cases of interest to us, the following properties are sufficient to characterise the radial Julia set.

- (1) $J_r(f) \subset J(f) \setminus I(f)$.
- (2) $J_r(f)$ is forward-invariant: $f(J_r(f)) \subset J_r(f)$. Furthermore, $J_r(f)$ is almost backwards-invariant except at critical values: if $z \in J(f)$ and $f(z) \in J_r(f)$, then either z is a critical point of f or $z \in J_r(f)$.
- (3) Every repelling periodic point and no parabolic periodic point belongs to $J_r(f)$.
- (4) Suppose that the forward orbit of $z \in J(f)$ has a finite accumulation point that is not in the closure of the union of the forward orbits of critical and asymptotic values. Then $z \in J_r(f)$.

The *fast escaping set* $A(f)$ plays a key role in our arguments. We use the definition given by Rippon and Stallard in [31], which is slightly different, but equivalent, to the original formulation from [6]. Let f be a transcendental entire function. For $r > 0$, define the maximum modulus

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

We also denote the n th iterate of the function $r \mapsto M(r, f)$ by $M^n(\cdot, f)$.

Since f is non-linear, we have

$$R(f) := \inf\{R \geq 0: M(r, f) > r \text{ for all } r \geq R\} < \infty. \quad (2.1)$$

For $R > R(f)$, define

$$A(f) := \{z \in \mathbb{C}: \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}. \quad (2.2)$$

It can be shown that the definition is independent of R . Again following [31], we also define

$$A_R(f) := \{z: |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

Attracting and parabolic exponential maps

Let us now turn to background results concerning exponential maps f_a as in (1.1). Recall that our main result concerns the case where $a \in F(f_a)$.

PROPOSITION 2.1 (Attracting and parabolic exponential maps). *Set $f_a(z) := e^z + a$, where $a \in \mathbb{C}$. Then $a \in F(f_a)$ if and only if f_a has an attracting or parabolic cycle.*

In this case, $F(f_a)$ is precisely the basin of attraction of this cycle, and ∞ is accessible from the connected component U_1 of $F(f_a)$ containing a , by an arc along which the real part tends to infinity.

Proof. This is well known; see [3, Section 9], particularly Corollary 2, Theorems 9 and 10 and the two paragraphs following Theorem 10. Accessibility is not mentioned explicitly, but

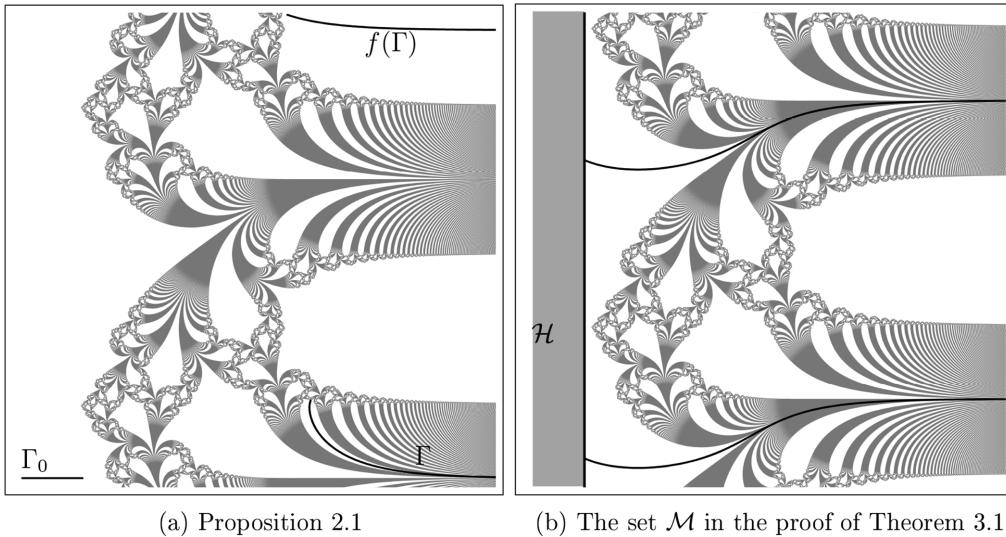


FIGURE 2. Illustration of the proofs of Proposition 2.1 and Theorem 3.1, using the exponential map $f = f_a$ from Figure 1b. As shown in (a), the curve Γ is constructed as a pullback of Γ_0 , which is a piece of the negative real axis. The set \mathcal{M} in (b) is the union of the left half-plane \mathcal{H} and the set $f^{-1}(\sigma)$, which is a union of infinitely many arcs to ∞ . (Here, σ is a subset of the curve Γ from (a).)

follows from the argument for unboundedness of Fatou components; compare [7, Section 2] or [32, Section 4.4]. In the case where f_a has an attracting fixed point, accessibility of infinity (and much more) can be found in the work of Devaney and Goldberg [10, Section 4]. As the proof of Proposition 2.1 will be instructive for our later constructions, and since we are not aware of a reference containing the statement in the precise form that we require, we provide the details for completeness.

For all $a \in \mathbb{C}$, the Fatou set $F(f_a)$ does not have wandering components [5, Theorem 12], nor components on which the iterates tend to infinity [5, Theorem 15]. Thus the only possible components of $F(f_a)$ are immediate attracting or parabolic basins, Siegel discs and their iterated preimages [5, Theorem 6]. Now suppose that $a \in F(f_a)$, so the forward orbit of a either converges to an attracting or periodic orbit, or is eventually contained in an invariant circle in a Siegel disc. In particular, the set of accumulation points of this orbit that belong to $J(f_a)$ is either empty or consists of a single parabolic cycle.

By [5, Theorem 7], the boundary of any Siegel disc is contained in the closure of the orbit of the omitted value a , and the immediate basin of any attracting or parabolic cycle must contain a . The former is impossible by the above, so we conclude that $F(f_a)$ consists of the basin of a single attracting or parabolic cycle.

Let p be the period of the component U_1 of $F(f_a)$ containing a . Consider the set $U_0 := f_a^{-1}(U_1)$; since U_1 contains a neighbourhood of a , U_0 contains a left half-plane \mathcal{H} . Since $f_a: U_0 \rightarrow U_1 \setminus \{a\}$ is a covering map, any connected component of U_0 intersects \mathcal{H} , and hence U_0 is connected. It follows that U_0 lies on the periodic cycle of Fatou components of U_1 , with $f_a^{p-1}(U_0) = U_1$. Set $n := \max(1, p-1)$, and let Γ be a connected component of $f_a^{-n}(\Gamma_0)$ contained in U_1 ; then Γ satisfies the properties claimed in the proposition (see Figure 2(a)). \square

In particular, we obtain the following description of the radial Julia set in the case where $a \in F(f_a)$ (as well as when f_a is postsingularly finite).

COROLLARY 2.2 (Radial Julia sets). *Let $f_a(z) = e^z + a$, $a \in \mathbb{C}$.*

- (a) *if f_a has an attracting periodic point, then $J_r(f_a) = J(f_a) \setminus I(f_a)$;*
- (b) *if f_a has a parabolic periodic point z_0 , then $J_r(f_a) = J(f_a) \setminus (I(f_a) \cup O^-(z_0))$, where $O^-(z_0)$ denotes the full backward orbit of z_0 ;*
- (c) *if the orbit of a under f_a is finite, then $J_r(f_a) = \mathbb{C} \setminus I(f_a)$.*

Proof. In each case, this follows from the properties (1)–(4) of $J_r(f_a)$ stated at the beginning of the section. Recall that always $J_r(f) \subset J(f) \setminus I(f)$ by (1). For the first two cases, recall the proof of Proposition 2.1. If f_a has an attracting periodic cycle, then the orbit of a is compactly contained in $F(f_a)$, and hence every non-escaping point belongs to the radial Julia set by (4). Now suppose that f_a has a parabolic periodic point z_0 ; then $O^-(z_0) \cap J_r(f_a) = \emptyset$ by (2) and (3). On the other hand, the orbit of a is in the Fatou set and accumulates only on the cycle of z_0 . Any orbit in $J(f_a) \setminus (I(f_a) \cup O^-(z_0))$ must accumulate at some finite point that is not on this cycle, and thus belongs to $J_r(f_a)$ by (4).

Finally, if the orbit of a is finite, then (similarly as in the first part of the proof of Proposition 2.1) $J(f_a) = \mathbb{C}$ and a eventually maps to a repelling periodic point z_0 . As in the parabolic case, we have $J(f_a) \setminus (I(f_a) \cup O^-(z_0)) \subset J_r(f_a)$, but here also $O^-(z_0) \subset J_r(f_a)$ by (2) and (3). So $J_r(f_a) = J(f_a) \setminus I(f_a) = \mathbb{C} \setminus I(f_a)$, as claimed. \square

Hairs and endpoints

The escaping set of any exponential map f_a decomposes into *hairs* (or *dynamic rays*) and (escaping) *endpoints* [33]. These concepts can be defined conveniently as follows; see [2, Definition 1.1].

DEFINITION 2.3 (Hairs and endpoints). Let $f_a(z) = e^z + a$, $a \in \mathbb{C}$. We say that a point $z_0 \in \mathbb{C}$ is on a *hair* if there exists an arc $\gamma: [-1, 1] \rightarrow I(f_a)$ such that $\gamma(0) = z_0$. A point $z_0 \in \mathbb{C}$ is an *endpoint* if z_0 is not on a hair and there is an arc $\gamma: [0, 1] \rightarrow \mathbb{C}$ such that $\gamma(0) = z_0$ and $\gamma(t) \in I(f_a)$ for all $t > 0$. We denote the set of endpoints by $E(f_a)$.

We recall the following properties of hairs and endpoints.

PROPOSITION 2.4 (Hairs and endpoints). *Let $f_a = e^z + a$, $a \in \mathbb{C}$.*

- (a) *Every point in $I(f_a)$ is on a hair or an endpoint.*
- (b) *If $z \in I(f_a)$ is on a hair, then $z \in A(f_a)$.*
- (c) *If $a \in F(f_a)$, then every point $z \in J(f_a)$ is on a hair or an endpoint.*
- (d) *If $a \in J(f_a)$, then there is a point $z \in J(f_a)$ that is neither on a hair nor an endpoint.*
- (e) *If a is on a hair or an endpoint, then $J(f_a) \setminus I(f_a)$ contains a dense collection of unbounded connected sets.*

Proof. The first claim follows from [33, Theorem 6.5]. The second follows by combining [33, Proposition 4.5] with well-known estimates on exponential growth (see Lemma 2.5) and the classification of path-connected components of $I(f_a)$ [14, Corollary 4.3]. Compare also [2, Section 4].

When f_a has an attracting periodic orbit, it was first shown in [7] that the Julia set of f_a is a ‘bouquet’ of hairs, where different hairs may share the same endpoint. This establishes (c) in this case. In [22, Corollary 9.3], the stronger statement is proved that $J(f_a)$ is a *pinched Cantor bouquet*. More precisely, the dynamics of f_a on its Julia set can be described as a quotient of that of $f_{\tilde{a}}$ on $J(f_{\tilde{a}})$, where $\tilde{a} \in (-\infty, -1)$, by an equivalence relation on the endpoints of $J(f_{\tilde{a}})$.

As stated in [22, Remark on p. 1967], these results also hold for exponential maps having a parabolic orbit, establishing (c). However, the details of the proof are omitted in [22]; they can be found in forthcoming work of M. Alhamd, which treats the general case of parabolic entire functions.

Finally, let us turn to (d) and (e), where $a \in J(f_a)$. If a is neither on a hair nor an endpoint, then there is nothing to prove. Otherwise, by [23], there exists a curve $\gamma: (0, \infty) \rightarrow I(f_a)$ with $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$, such that the closure $\hat{\gamma}$ of γ in $\hat{\mathbb{C}}$ is an indecomposable continuum. Moreover, the construction ensures that $\hat{\gamma}$ does not separate the sphere, that $\hat{\gamma} \cap I(f_a) = \gamma$ and that $a \notin \hat{\gamma}$.

As an indecomposable continuum, $\hat{\gamma}$ has uncountably many composants [20, Theorem 11.15], each of which is dense in $\hat{\gamma}$. Exactly one of these components contains γ , and by the above no other composant intersects the escaping set. Thus every other composant is an unbounded connected subset of $J(f_a) \setminus I(f_a)$.

By [18], the union of composants of $\hat{\gamma}$ that contain a point accessible via a curve in $\hat{\mathbb{C}} \setminus \hat{\gamma}$ is of first category in $\hat{\gamma}$. In particular, $\hat{\gamma}$ contains a point that is not accessible from $I(f_a)$, proving (d). (It also follows directly from the construction in [23] that no point of $\hat{\gamma}$ is an endpoint.)

Moreover, let $\tilde{\gamma}$ be a connected component of $f_a^{-1}(\gamma)$; then it follows that $\tilde{\gamma}$ has the same properties as γ . Since iterated preimages of γ are dense in $J(f_a)$, and unbounded connected sets of non-escaping points are dense in the closure of each preimage, the proof is complete. \square

Growth of exponential maps

It is well known (compare [33, Lemma 2.4]) that all exponential maps (and indeed all entire functions of finite order and positive lower order) share the same maximal order of growth of their iterates, namely iterated exponential growth. Hence we can use a single model function from this class to describe maximal growth rates of exponential maps. For this purpose, it has become customary to use

$$F: [0, \infty) \rightarrow [0, \infty); \quad t \mapsto e^t - 1. \quad (2.3)$$

We will use the following elementary fact; compare, for example, inequalities (10.1) and (10.2) in [22].

LEMMA 2.5 (Iterated exponential growth). *Fix $K \geq 1$ and $a \in \mathbb{C}$. Then*

$$F(\operatorname{Re} z - 1) + K \leq |f_a(z)| \leq F(\operatorname{Re} z + 1) - K \quad (2.4)$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq \ln(1 + 2(|a| + K))$.

Furthermore, for all $R \geq \max(3, \ln(1 + 2(|a| + K)))$ and all $n \geq 1$,

$$R < F^n(R - 1) + K \leq M^n(R, f_a) \leq F^n(R + 1) - K.$$

Proof. To prove (2.4), set $r := \operatorname{Re} z$. Then

$$\begin{aligned} F(r - 1) + K &< e^{r-1} + K = \frac{1}{e} e^r + K \leq \frac{1}{2} e^r + K \\ &= e^r - \frac{e^r}{2} + K \leq e^r - |a| = |e^z| - |a| \leq |e^z + a| = |f_a(z)| \end{aligned}$$

by assumption on r . Similarly,

$$F(r + 1) - K = e \cdot e^r - K - 1 \geq 2e^r - K - 1 \geq e^r + |a| \geq |f_a(z)|.$$

Now let R be as in the second claim. Then, by (2.4),

$$F(R + 1) - K \geq M(R, f_a) \geq |f_a(R)| \geq F(R - 1) + K \geq F(R - 1) > R, \quad (2.5)$$

where we use the fact that $e^{R-1} > R + 1$ for $R \geq 3$. Applying (2.5) inductively, we see that

$$M^{n+1}(R, f_a) + K \leq F(M^n(R, f_a) + 1) \leq F(M^n(R, f_a) + K) \leq \dots \leq F^{n+1}(R + 1),$$

and analogously for the lower bound. \square

REMARK 2.6. It follows, in particular, that $R(f_a) \leq R$ for R as in the statement of the lemma. (Recall that $R(f_a)$ was defined in (2.1).) However, in fact

$$R(f_a) = 0 \tag{2.6}$$

for all $a \in \mathbb{C}$, which means that $A_R(f_a)$ is defined for all $R > 0$, and simplifies our statements in the following.

Indeed, if $R + e^{-R} < -\operatorname{Re} a$, then

$$M(R, f_a) \geq -\operatorname{Re} f_a(-R) = -\operatorname{Re} a - e^{-R} > R.$$

On the other hand, if $R + e^{-R} \geq -\operatorname{Re} a$, then

$$M(R, f_a) \geq \operatorname{Re} f_a(R) = e^R + \operatorname{Re} a \geq e^R - R - e^{-R} = 2 \sinh(R) - R.$$

Since $\sinh(R) > R$ for $R > 0$, we have $M(R, f_a) > R$ in this case also, as claimed.

A simple consequence of Lemma 2.5, which is crucial to our proof of Theorem 1.3, is the following. Suppose that a starting point z has large real part; then we know by the above that $|f_a(z)|$ is very large. If we know that $|\operatorname{Im} f_a(z)|$ and $-\operatorname{Re} f_a(z)$ are comparatively small compared to this value, then clearly $\operatorname{Re} f_a(z)$ is again very large and positive. Under suitable hypotheses, we can continue inductively and conclude that z escapes (quickly) to infinity. Again, this is a well-known argument in the study of exponential maps; compare, for example, [33, Proof of Theorem 4.4]. We will use it in the following form.

COROLLARY 2.7 (Continued growth). *Let $a \in \mathbb{C}$ and $\mu \geq 0$. Then there is $K \geq \mu + 2$ such that the following holds for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq K$. If $n \geq 0$ is such that*

$$\max(-\operatorname{Re} f_a^k(z), |\operatorname{Im} f_a^k(z)|) \leq F^k(\mu) \tag{2.7}$$

for $0 \leq k < n$, then

$$|f_a^n(z)| \geq F^n(\operatorname{Re} z - 2).$$

Proof. Set $K := \max\{2 + \ln(5 + |a|), \mu + 2\}$. Suppose that $r := \operatorname{Re} z \geq K$, and that n is as in the statement of the corollary. Observe that the claim is trivial for $n = 0$. We will prove, by induction on $n \geq 1$, the stronger claim

$$|f_a^n(z)| \geq F^n(r - 2) + F^n(\mu) + 2. \tag{2.8}$$

Let $n \geq 1$, and suppose that the claim holds for smaller values of n . Then

$$\operatorname{Re} f_a^{n-1}(z) \geq F^{n-1}(r - 2) + 2 \geq r \geq K.$$

This is true trivially for $n = 1$, and by the inductive hypothesis (2.8) and assumption (2.7) for $n > 1$. By Lemma 2.5,

$$|f_a^n(z)| \geq F(\operatorname{Re} f_a^{n-1}(z) - 1) + 2 \geq 2F^n(r - 2) + 2 \geq F^n(r - 2) + F^n(\mu) + 2,$$

as claimed. \square

Separation

Recall that two points $a, b \in X$ are *separated* in a metric space X if there is a closed and open ('clopen') set $U \subset X$ that contains a but not b . We will use the following simple lemma only in the case where $X \subset \hat{\mathbb{C}}$ and $x = \infty$.

LEMMA 2.8. *Let X be a metric space and $x \in X$. Suppose that $A := X \setminus \{x\}$ is totally separated. Assume furthermore that every point of A is separated from x in X . Then X is totally separated.*

Proof. Let $a, b \in X$ with $a \neq b$. If either a or b is equal to x , then by assumption a and b are separated in X . Otherwise, let U be a clopen set in A containing a but not b , and let V be a clopen set in X containing a but not x . Then U is open, but not necessarily closed, in X .

Set $W := U \cap V$; then W is open in X and closed in A . Furthermore, $x \notin V = \overline{V} \supset \overline{W}$, and hence W is also closed in X . So W is a clopen set of X containing a but not b , and the proof is complete. \square

Separation and spiders' webs

We will use the following notion.

DEFINITION 2.9 (Separation in $\hat{\mathbb{C}}$). If $x, y \in \hat{\mathbb{C}}$, we say that $E \subset \hat{\mathbb{C}}$ *separates* x from y if x and y are separated in $(\mathbb{C} \setminus E) \cup \{x, y\}$.

Analogously, E separates a set $X \subset \hat{\mathbb{C}}$ from a point $y \in \hat{\mathbb{C}}$ if there is a clopen set $U \subset (\mathbb{C} \setminus E) \cup X \cup \{y\}$ containing X but not y .

We now prove Theorem 1.5, in the following slightly more precise version.

THEOREM 2.10 (Characterisation of spiders' webs). *Let $E \subset \mathbb{C}$ (connected or otherwise). The following are equivalent.*

- (a) *There is a sequence of domains G_n as in the definition of a spider's web.*
- (b) *E separates every compact set in \mathbb{C} from ∞ .*
- (c) *E separates every finite point z from ∞ .*

Suppose now that one of these equivalent conditions holds. Then the following are equivalent:

- (1) *E is connected (that is, E is a spider's web);*
- (2) *there is a dense collection of unbounded connected subsets of E ;*
- (3) *$E \cup \{\infty\}$ is connected.*

Proof. Clearly, (b) implies (c). Furthermore, (a) implies (b). Indeed, let G_n be the domains from the definition of a spider's web. If $X \subset \mathbb{C}$ is compact and n is sufficiently large that $X \subset G_n$, then $(G_n \setminus E) \cup X$ is a clopen subset of $(\mathbb{C} \setminus E) \cup X \cup \{\infty\}$, as required.

Now suppose that (b) holds. We claim that for every non-empty, compact and connected $K \subset \mathbb{C}$, there is a bounded simply connected domain $G = G(K)$ with $K \subset G$ and $\partial G \subset E$.

Indeed, as E separates K and ∞ , by definition there is a relatively closed and open subset

$$U' \subset A := (\mathbb{C} \setminus E) \cup K \cup \{\infty\}$$

such that $K \subset U' \subset \mathbb{C}$. Let $U \subset \hat{\mathbb{C}}$ be open such that $U' = U \cap A$. Since U' is relatively closed in A , we can choose U such that U is bounded and $\partial U \subset \hat{\mathbb{C}} \setminus A \subset E$.

Now let V be the connected component of U containing K , and let $G = G(K)$ be the fill of V . (That is, G consists of V together with all bounded complementary components.) Clearly $\partial G \subset \partial U \subset E$.

So we can define a sequence of simply connected domains by letting K_0 be the disc $\overline{D(0, 1)}$, and defining inductively $G_j := G(K_j)$ and $K_{j+1} := \overline{D(0, j)} \cup \overline{G_j}$. The domains G_j satisfy the requirements in the definition of a spider's web, so (a) holds.

Finally, suppose (c) holds. Let $K \subset \mathbb{C}$ be a compact set. Then for every $x \in K$ there is a bounded open set $U \subset \mathbb{C}$ such that $x \in U$ and $\partial U \subset E$.

Since K is compact, there are $k \in \mathbb{N}$ and U_1, \dots, U_k as above such that $K \subset U := \bigcup U_j$. Clearly

$$\partial U \subset \bigcup \partial U_j \subset E,$$

and U is bounded. So ∂U separates K from ∞ .

This completes the proof of the equivalence of the three conditions (a) to (c). For the final statement, first observe that (1) \Rightarrow (2) \Rightarrow (3) for all unbounded sets E .

Clearly the opposite implications do not hold in general, so suppose now that (a) holds, and that E is disconnected. We must show that $E \cup \{\infty\}$ is also disconnected.

Let (G_n) be the sequence of domains from (a). If U and V are disjoint non-empty clopen subsets of E with $U \cup V = E$, then at least one of these sets, say U , must contain ∂G_n for infinitely many n . Choose such n sufficiently large that also $V \cap G_n \neq \emptyset$; then it follows that $V \cap G_n$ is a bounded clopen subset of E , and hence is also a non-empty and non-trivial proper clopen subset of $E \cup \{\infty\}$. \square

REMARK 2.11. In (c), it is crucial to require separation for all finite points, not just those in $\hat{\mathbb{C}} \setminus E$. That is, E being a spider's web is a stronger condition than requiring that the quasicomponent of ∞ in $\hat{\mathbb{C}} \setminus E$ is a singleton. (Recall that the quasicomponent of a point x in a metric space X consists of all points of X not separated from x .)

This is true even in the case where $\hat{\mathbb{C}} \setminus E$ is totally separated. Indeed, let $A \subset \hat{\mathbb{C}}$ be a connected set containing 0 and ∞ , and having an explosion point at 0. Then $E := \{0\} \cup \hat{\mathbb{C}} \setminus A$ is not a spider's web, as 0 is not separated from infinity in $A = (\hat{\mathbb{C}} \setminus E) \cup \{0\}$, but $\hat{\mathbb{C}} \setminus E = A \setminus \{0\}$ is totally separated.

REMARK 2.12. Suppose that E is forward-invariant by a transcendental entire function f . Then, by the blowing-up property of the Julia set, (a)–(c) above are equivalent to

(d) E separates some finite point $z \in J(f)$ from ∞ .

Similarly, if E is backward-invariant, then (3) is equivalent to

(4) there is $E' \subset E$ such that $E' \cup \{\infty\}$ is connected; furthermore, if U is a Fatou component, then no point of $U \cap E$ is separated from $\partial U \cap E$ in E .

(Indeed, if (4) holds, then $f^{-n}(E') \cup \{\infty\}$ is connected for all n [2, Lemma 4.3], and these sets are dense in $J(f)$.)

In particular, a completely invariant set E is a spider's web if and only if (d) and (4) hold. See [34, Theorem 1.5].

3. The exponential family

Theorem 1.3 is a straightforward consequence of the following result. (Recall Definition 2.9.)

THEOREM 3.1 (Separation using fast escaping points). *Let $f_a(z) = e^z + a$, and assume that $a \in F(f_a)$. Then, for all $R > 0$ and all $z_0 \in \mathbb{C}$, z_0 is separated from infinity by $A_R(f_a) \cup F(f_a)$.*

Proof. Let U_1 be the component of $F(f_a)$ containing a , and let $\varepsilon = e^{-c}$ be small enough such that $D := \overline{D(a, \varepsilon)} \subset U_1$. By Proposition 2.1, there is an arc $\sigma \subset U_1$ connecting D to ∞ , intersecting D only in the finite endpoint, and along which real parts tend to infinity.

Consider the closed set

$$\mathcal{M} := f_a^{-1}(D \cup \sigma).$$

(See Figure 2(b).) Then \mathcal{M} consists of the closed left half-plane $\mathcal{H} := \{z : \operatorname{Re}(z) \leq -c\}$, together with countably many arcs connecting this half-plane to infinity. Each of these arcs is a component of $f_a^{-1}(\sigma)$, and hence they are all $2\pi i\mathbb{Z}$ -translates of each other. Furthermore, each of these preimage components is bounded in the imaginary direction (since the argument is bounded along σ).

Let $(S_j)_{j=-\infty}^{\infty}$ denote the complementary components of \mathcal{M} , and set

$$\delta := \sup_{z, w \in S_0} |\operatorname{Im} z - \operatorname{Im} w| = \sup_{z, w \in S_j} |\operatorname{Im} z - \operatorname{Im} w| \text{ for all } j \in \mathbb{Z}.$$

Now let $R > 0$. Note that it follows from (2.6) that $M(r, f_a) > r$, for $r > 0$. Moreover, for $0 < R' < R$ we have $A_R(f_a) \cup F(f_a) \subset A_{R'}(f_a) \cup F(f_a)$. Hence we can increase R , if necessary, to ensure that

$$R > \max(|z_0|, c, 3, \ln(1 + 2(|a| + \delta))).$$

Set $D_k := D(0, M^k(R, f_a))$ for $k \geq 0$, and consider the set

$$X := \bigcap_{k \geq 0} \left(f_a^{-k}(\mathbb{C} \setminus D_k) \cup \bigcup_{j=0}^{k-1} f_a^{-j}(\mathcal{M}) \right).$$

That is, if $x \in X$ and $k \geq 0$, then either $|f_a^k(z)| \geq M^k(R, f_a)$, or the orbit of z has entered \mathcal{M} before time k . Note that since $|z_0| < R$ and $X \subset \mathbb{C} \setminus D(0, R)$ we have $z_0 \notin X$. Also $X \subset A_R(f_a) \cup F(f_a)$ by definition. It is thus enough to show that z_0 is separated from ∞ by X . Since X is closed as an intersection of closed sets, this is equivalent to showing that the connected component V of $\mathbb{C} \setminus X$ containing z_0 is bounded.

By the definition of X , the modulus of the forward images of any point in V must fall behind the growth given by $M^k(R, f_a)$ in order to be able to enter \mathcal{M} ; that is,

$$V \cap f_a^{-n}(\mathcal{M}) \subset (\mathbb{C} \setminus X) \cap f_a^{-n}(\mathcal{M}) \subset \bigcup_{k=0}^n f_a^{-k}(D_k)$$

for all $n \geq 0$. Since $f_a(D_k) \subset D_{k+1}$, we have

$$f_a^n(V) \cap \mathcal{M} \subset D_n \tag{3.1}$$

for all $n \geq 0$ (see Figure 3).

Let $n \geq 0$. Since $|z_0| < R$, we have $f_a^n(z_0) \in f_a^n(V) \cap D_n$. If S_j is a complementary component of \mathcal{M} that does not intersect D_n , then $f_a^n(z_0) \in f_a^n(V) \setminus S_j$, and furthermore we have $f_a^n(V) \cap \partial S_j = f_a^n(V) \cap \mathcal{M} \cap \overline{S_j} = \emptyset$ by (3.1). Hence $S_j \cap f_a^n(V) = \emptyset$ (see Figure 3). Thus (3.1) can be reformulated as

$$f_a^n(V) \subset D_n \cup \bigcup \{S_j : S_j \cap D_n \neq \emptyset\}. \tag{3.2}$$

Now let $z \in V$. Then there is a minimal $N \geq 0$ such that $f_a^N(z) \in D_N$. By (3.1) we have $f_a^n(z) \notin \mathcal{M}$ for $n < N$, and hence

$$\operatorname{Re} f_a^n(z) > -c > -R > -F^n(R+1). \tag{3.3}$$

Moreover, using Lemma 2.5 and choice of R , we see from (3.2) that

$$|\operatorname{Im} f_a^n(z)| \leq M^n(R, f_a) + \delta \leq F^n(R+1) \tag{3.4}$$

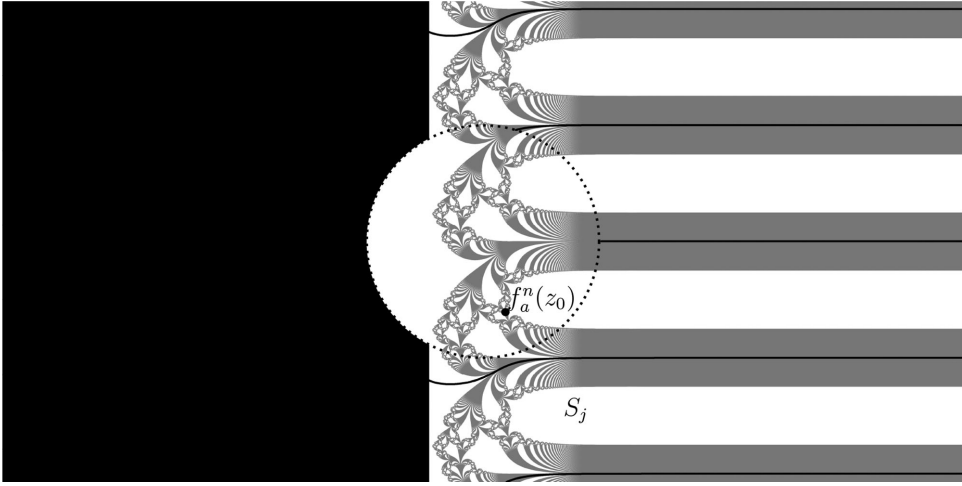


FIGURE 3. Illustration of the proof of Theorem 3.1. The domain $f_a^n(V)$ cannot intersect the set shown in black, which is the part of M from Figure 2(b) that does not lie in the disc $D_n = D(0, M^n(f_a, R))$. (The boundary of this disc is shown as a dotted line.) Since $f_a^n(z_0) \in D_n \cap f_a^n(V)$, any strip S_j which does not intersect D_n also cannot intersect $f_a^n(V)$.

for all n . Hence

$$\operatorname{Re} z \leq \max(R + 3, K), \quad (3.5)$$

where K is as in Corollary 2.7, applied to $\mu = R + 1$. Indeed, otherwise we could conclude from Corollary 2.7 that

$$|f_a^{N+1}(z)| \geq F^{N+1}(\operatorname{Re} z - 2) > F^{N+1}(R + 1),$$

which contradicts the fact that $|f_a^{N+1}(z)| < M^{N+1}(R, f_a) < F^{N+1}(R + 1)$ by choice of N and Lemma 2.5.

In conclusion, $\operatorname{Im} z$ is bounded in V by (3.4), while $\operatorname{Re} z$ is bounded from below by (3.3) and from above by (3.5). So V is bounded, as required. \square

Theorem 3.1 implies that $A_R(f_a) \cup F(f_a)$ has the structure of a spider's web.

COROLLARY 3.2 (Spiders' webs). *Let $f_a(z) = e^z + a$, where $a \in F(f_a)$. Then, for all $R > 0$, $A_R(f_a) \cup F(f_a)$ is a spider's web.*

Proof. Let $E := A_R(f_a) \cup F(f_a)$. By Theorem 3.1, property (c) in Theorem 2.10 is satisfied for E . Since $J(f_a)$ is nowhere dense, $F(f_a)$ is dense in E . Moreover, it follows from Proposition 2.1 that every connected component of $F(f_a)$ contains an arc along which the real part tends to infinity, and so is unbounded. Hence it follows from the second part of Theorem 2.10 that E is a spider's web. \square

We can also deduce Theorem 1.3, in the following more precise version.

COROLLARY 3.3 (Total separation). *Let $f_a(z) = e^z + a$, where $a \in F(f_a)$. Then the set $(E(f_a) \setminus A_R(f_a)) \cup \{\infty\}$ is totally separated for all $R \geq 0$.*

In particular, $J_m(f_a) \cup \{\infty\}$ is totally separated.

Proof. The set $E(f_a) \subset J(f_a)$ is totally separated by [2, Theorem 1.7]. Hence the first claim follows from Theorem 3.1 and Lemma 2.8.

By Proposition 2.4, the set $A(f_a)$ contains all points on hairs (as well as some endpoints). Hence

$$J_m(f_a) = J(f_a) \setminus A(f_a) = E(f_a) \setminus A(f_a) \subset E(f_a) \setminus A_R(f_a),$$

and thus $J_m(f_a) \cup \{\infty\}$ is totally separated by the first claim. \square

REMARK 3.4. On the other hand, it follows from the construction in [2, Remark 4.6] that the connected component of ∞ in $(E(f_a) \cap A_R(f_a)) \cup \{\infty\}$ is non-trivial for all R .

4. Exponential maps whose singular value lies in the Julia set

In this section we remark upon the case where $a \in J(f_a)$. In this case, $F(f_a)$ is either empty or consists of a cycle of Siegel discs, together with their preimages. We can still apply our method of proof from the previous section to obtain the following result.

THEOREM 4.1 (Points staying away from the singular value). *Let $a \in \mathbb{C}$ with $a \in J(f_a)$, let $\varepsilon > 0$ and $R > 0$. Let S denote the set of points $z \in \mathbb{C}$ with*

$$\inf_{n \geq 0} |f_a^n(z) - a| > \varepsilon. \quad (4.1)$$

Then every point $z_0 \in S$ is separated from infinity by $A_R(f_a) \cup (\mathbb{C} \setminus S)$.

REMARK. If $a \in F(f_a)$, then there exists $\varepsilon > 0$ such that every point in $J(f_a)$ trivially satisfies 4.1. Hence, by Theorem 3.1, we can remove the hypothesis ‘ $a \in J(f_a)$ ’ in Theorem 4.1 if we replace ‘ $z \in \mathbb{C}$ ’ by ‘ $z \in J(f_a)$ ’ (or, even, ‘ z not belonging to an attracting or parabolic basin’).

Proof. Write $\varepsilon = e^{-c}$ and $D := \overline{D(a, \varepsilon)}$. Since D intersects the Julia set, it follows by the blowing-up property of the Julia set (see, for example, [5, Section 2; 30, Lemma 2.1]) that there is a point $\zeta \in D$ and $n \geq 1$ such that $f_a^n(\zeta) \in \mathcal{H} = \{z : \operatorname{Re}(z) \leq -c\}$. Now connect $f_a^n(\zeta)$ to infinity by an arc σ_0 in \mathcal{H} , chosen such that σ_0 avoids $f_a^k(a)$ for $k = 0, \dots, n-1$. As in Proposition 2.1, the connected component σ_1 of $f_a^{-n}(\sigma_0)$ containing ζ is an arc in $\mathbb{C} \setminus S$ whose real parts tend to $+\infty$. By deleting the maximal piece of σ_1 connecting ζ to ∂D , we obtain an arc σ connecting D to infinity. Set $\mathcal{M} := f_a^{-1}(D \cup \sigma)$, and continue as in the proof of Theorem 3.1.

So we again obtain a closed set X , and see that the connected component V of $\mathbb{C} \setminus X$ containing z_0 is bounded. All points in the set X either belong to $A_R(f_a)$, or otherwise map into the left half-plane $\mathcal{H} = \{z : \operatorname{Re}(z) \leq -c\}$, and thus into D . So $X \subset A(f_a) \cup (\mathbb{C} \setminus S)$, and we obtain the desired conclusion. \square

In the case where the Fatou set is non-empty, Theorem 4.1 immediately implies the following result, proved in [21].

THEOREM 4.2. *Let $f_a(z) = e^z + a$, and assume that f_a has a cycle $U_1 \mapsto \dots \mapsto U_n \mapsto U_1$ ($n \geq 1$) of Siegel discs such that no ∂U_j contains the singular value a .*

Then all U_j are bounded.

(The proof of Theorem 4.2 given in [21] also relies on using a similar set as \mathcal{M} above; compare [21, Figure 2].)

To conclude the section, note that it follows immediately from Proposition 2.4(e) that, when $a \in J(f_a)$, the structure of the full set of non-escaping points in $J(f_a)$ can look very different from the case where $a \in F(f_a)$. (See [34, Theorem 1.3] for a stronger result obtained by Sixsmith since our paper was completed.)

COROLLARY 4.3 (Large meandering sets). *Let $a \in \mathbb{C}$ be such that, for the exponential map f_a , the omitted value a is either an endpoint or on a hair. Then $(J(f_a) \setminus I(f_a)) \cup \{\infty\}$ is connected. In particular, neither $I(f_a)$ nor $A(f_a)$ is a spider's web.*

Recall that, when f_a is postsingularly finite, the radial Julia set coincides with the set of non-escaping points (Corollary 2.2(c)). Thus we obtain, in particular, the statement concerning $J_r(f_a)$ made in the introduction.

5. Fatou's function

We now turn to studying Fatou's function

$$f: \mathbb{C} \rightarrow \mathbb{C}; \quad z \mapsto z + 1 + e^{-z}. \quad (5.1)$$

Observe that f is semiconjugate to

$$h: \mathbb{C} \rightarrow \mathbb{C}; \quad \zeta \mapsto e^{-1}\zeta e^{-\zeta} \quad (5.2)$$

via the correspondence $\zeta = g(z) := \exp(-z)$.

It is well known that $J(f)$ is a Cantor bouquet while $F(f)$ consists of a single domain in which the iterates tend to infinity; see, for example, [26, Theorem 1.3]. Hence it again makes sense to speak about *hairs* and *endpoints* of $J(f)$. Moreover, also by [26, Theorem 1.3], all points on hairs belong to $A(f)$; in other words, all points in $J_m(f)$ are endpoints. We refer to [12] for further background.

As noted in the introduction, the following implies [12, Theorems 1.1 and 5.2].

THEOREM 5.1 (Fatou's web revisited). *Let f be Fatou's function (5.1). Then the set $F(f) \cup A(f)$ is a spider's web, and its complement $J_m(f)$, that is, the set of meandering endpoints of f , together with infinity forms a totally separated set.*

We could prove this theorem by mimicking the proof of Theorem 1.2. Instead, let us show that the latter in fact implies Theorem 5.1, using the semiconjugacy between f and h , together with known (albeit non-elementary) results.

PROPOSITION 5.2 (Structure of h). *Let h be as in (5.2), and let f_{-2} be the exponential map $z \mapsto e^z - 2$. Then there is a homeomorphism $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ such that $\varphi(J(f_{-2})) = J(h)$, $\varphi(I(f_{-2})) = I(h)$ and $\varphi(A(f_{-2})) = A(h)$.*

In particular, $J_m(h) \cup \{\infty\}$ is totally separated, and the complement of $J_m(h)$ is a spider's web.

Proof. The function h is conjugate to $\tilde{h}: w \mapsto (w+1)e^w - 1$, via $w = -\zeta - 1$, so it is enough to prove the claim for \tilde{h} . It is shown in [24] that there is a quasiconformal homeomorphism $\tilde{\varphi}$ that conjugates f_{-2} and \tilde{h} on their Julia sets (see [24, Figure 1]); in particular, $\tilde{\varphi}(J(f_{-2})) = J(\tilde{h})$ and $\tilde{\varphi}(I(f_{-2})) = I(\tilde{h})$. Since quasiconformal maps are Hölder, it also follows that $\tilde{\varphi}(A(f_{-2})) = A(\tilde{h})$.

Indeed, recall from Lemma 2.5 that $z \in A(f_{-2})$ if and only if for some, and hence all, $T > 0$ there is $n_0 \geq 0$ such that $|f_{-2}^{n_0+k}(z)| \geq F^k(T)$ for all $k \geq 0$. By a similar calculation, the same

is true for h and, in fact, any entire function of positive lower order and finite upper order (compare, for example, [26, Lemma 3.4]).

Now let $z \in I(f_{-2})$ and $w := \tilde{\varphi}(z) \in I(\tilde{h})$, and denote the orbits of these points under the corresponding maps by $(z_n)_{n \geq 0}$ and $(w_n)_{n \geq 0}$, respectively. Since $\tilde{\varphi}$ is Hölder, there is $\alpha > 1$ such that

$$|w_n|^{1/\alpha} \leq |z_n| \leq |w_n|^\alpha.$$

By an elementary calculation, $F(T^\alpha) > F(T)^\alpha$ for all sufficiently large T . It follows that $z \in A(f_{-2})$ if and only if $w \in A(\tilde{h})$, as claimed.

Recall that $J_m(E) = J(E) \setminus A(E)$ by definition, for any entire function E ; so also $\tilde{\varphi}(J_m(f_{-2})) = J_m(\tilde{h})$. The final claim now follows from Corollaries 3.2 and 3.3. \square

Proof of Theorem 5.1. Let $g: z \mapsto \exp(-z)$ be the semiconjugacy between f and h , so $g \circ f = h \circ g$. By [6, Theorems 1 and 5], we have $g(J(f)) \subset J(h)$ and $g^{-1}(A(h)) \subset A(f)$. (It is well known that actually $g(J(f)) = J(h)$, which also follows from [6] and the fact that $A(f) \subset J(f)$, but we do not require this here.)

Hence

$$J_m(f) = J(f) \setminus A(f) \subset g^{-1}(J(h) \setminus A(h)) = g^{-1}(J_m(h)).$$

Since $h(x) = x \cdot e^{-(x+1)} < x$ for $x > 0$, we see that $[0, \infty) \in F(h)$. Taking inverse branches of g on the slit plane $\mathbb{C} \setminus [0, \infty)$, we therefore see that $J_m(f)$ is contained in a countable collection of homeomorphic copies of $J_m(h)$, which are mutually separated from each other by the horizontal lines whose imaginary parts are even multiples of π .

Since $A(h) \cup F(h)$ is a spider's web (Proposition 5.2) it follows from Theorem 2.10 that it separates every finite point from ∞ . Hence the above argument implies that $A(f) \cup F(f)$ separates every finite point from ∞ and since it is connected it is also a spider's web. \square

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